CONDITIONS FOR NONLINEAR STABILITY OF FLOWS OF AN IDEAL INCOMPRESSIBLE LIQUID

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Stability of steady state flows of an ideal incompressible liquid with homogeneous density with some type of symmetry (translational, axial, rotational, or helical) is considered. Two types of sufficient conditions for nonlinear stability are obtained, which can be proven by constructing two types of functionals which have absolute minima at the given steady state solutions. Each of the functionals used is the sum of the kinetic energy and some other integral, specific to the given class of motion. The first type of stability conditions are a generalization to the case of finite perturbations and a new class of flows of the well known Rayleigh criterion [1] for "centrifugal" stability of rotating flows relative to perturbations with rotational symmetry. In the same sense the second type of stability conditions generalize another result, also originally proposed by Rayleigh, according to which plane-parallel flow of a liquid is stable in the absence of an inflection point in the velocity profile [1]. A nonlinear variant of the latter condition for the class of planar motions was first obtained in [2]. To systematize the results extensive use is made of the analogy between the effects of density stratification and rotation in the form of [3]. The results to be presented relate to stability of a wide class of hydrodynamic flows having the required symmetry. For example, they relate to flows in tubes and channels which rotate or are at rest, and flows with concentrated annular or spiral vortices.

1. Flows with Helical Symmetry. We will consider nonsteady state motions of an ideal incompressible liquid with homogeneous density. In the cylindrical coordinate system ϕ, r, z the components of the velocity field are u, v, w; p is the pressure field. For the motions studied with helical symmetry u, v, w, and p are functions of three independent variables: r, $\mu \equiv a\phi - bz$, and time t. For example,

$$p = p(r, \mu, t). \tag{1.1}$$

The term b denotes any real number; without limiting generality the parameter a may be assumed to have one of only two values: 0 and 1. At a = 1 all solutions of the form of Eq. (1.1) will be periodic in μ with period 2π and it will be sufficient to consider μ values in the interval

$$0 \leqslant \mu \leqslant 2\pi. \tag{1.2}$$

At a = 0 (rotational symmetry) solutions may not be periodic. Using the notation of [3]

$$\begin{gathered}\alpha \equiv au - brw,\ \beta \equiv bru + aw,\\ R \equiv a^2 + b^2r^2,\ g \equiv b^2r/R^2,\ K \equiv 2ab/R^2,\end{gathered} \tag{1.3}$$

the basic equations of motion for solutions of the form of Eq. (1.1) transform to

$$\begin{gathered}D(r\alpha/R) + K\beta rv = -p_\mu,\\ Dv - K\beta\alpha - (a\alpha/R)^2/r = -p_r + g\beta^2,\\ D\beta = 0,\ v_r + v/r + \alpha_\mu/r = 0,\\ D \equiv \frac{\partial}{\partial t} + v\frac{\partial}{\partial r} + \frac{\alpha}{r}\frac{\partial}{\partial \mu}.\end{gathered} \tag{1.4}$$

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Subscripts on the independent variables denote the corresponding partial derivatives.

Differentiating the first expression of Eq. (1.4) with respect to r, the second with respect to μ, and subtracting the one from the other, we obtain

$$D\omega + \left(v\frac{\partial}{\partial r} + \frac{\alpha}{r}\frac{\partial}{\partial\mu}\right)\frac{2ab\beta}{R^2} + \frac{b^2}{R^2}(\beta^2)_\mu = 0, \tag{1.5}$$

where $\omega \equiv (1/r)[(r\alpha/R)_r - v_\mu]$.

If the motion of Eq. (1.1) takes place in a fixed region, then its boundaries must possess the required symmetry, i.e., the function of two variables

$$f(r, \mu) = 0 \tag{1.6}$$

must be specified. Conditions for nonpenetration at Eq. (1.6) for the true velocity components u, v, w, written in terms of Eqs. (1.3), (1.6), give

$$vf_r + f_\mu\alpha/r = 0. \tag{1.7}$$

The flow region will be considered either single- or double-bounded; in the latter case the boundary Eq. (1.6) will consist of two components – internal and external.

Helical wall geometry in Eq. (1.6) may appear far-fetched, but helical tubes are indeed widely used in heat exchange apparatus [4]. At the same time an important special case of Eq. (1.6) is a circular cylindrical boundary with flow region

$$R_1 < r < R_2. \tag{1.8}$$

At $R_1 \neq 0$ Eq. (1.8) corresponds to flow between two concentric cylinders, and at $R_1 = 0$ to flow in a circular tube.

At a = 1 Eqs. (1.4), (1.5) and boundary conditions (1.7) can be considered conveniently in the plane of the polar coordinates r, μ with radial coordinate r and angular coordinate μ, Eq. (1.2). In this plane the closed curves of Eq. (1.6) bound the flow region τ. With such an approach Eqs. (1.4)-(1.6) prove quite similar to the equations and boundary conditions for planar flows of a liquid with inhomogeneous density (stratified) [3]. The role of μ, the velocity components, is played by the quantity α, and the role of density by the variable β (or β^2). The corresponding "mass force field" is directed along the radius from the center and has a value g, Eq. (1.3). For motions with rotational symmetry (a = 0) this similarity becomes equivalence [3].

For the general case of Eqs. (1.4)-(1.7), the similarity of Eq. (1.1) to the motion of an inhomogeneous liquid is so fargoing that there exists an energy integral E in the form of the sum of fictitious "kinetic" T and "potential" P energies:

$$E = T + P, \quad T \equiv \frac{1}{2}\int_\tau \left(\frac{\alpha^2}{R} + v^2\right)d\tau \quad (d\tau \equiv r\,dr\,d\mu), \tag{1.9}$$

$$P \equiv \int_\tau \beta^2 U\,d\tau, \quad U = U(r) \equiv \int_0^r g(\xi)\,d\xi.$$

In terms of the original velocity comonents u, v, w, the quantity E is the kinetic energy taken over one period, Eq. (1.2). The other integral of Eqs. (1.4)-(1.7) is defined in terms of an arbitrary function Φ(β):

$$I = \int_\tau \Phi(\beta)\,d\tau = \text{const}. \tag{1.10}$$

2. Analogy of Hydrostatic Equilibrium States. The problem of Eqs. (1.4)-(1.7) has exact solutions – "states of rest":

$$\alpha = v = 0, \ \beta = \beta_0(r), \tag{2.1}$$

containing one arbitrary function $\beta_0(r)$. In terms of the true velocity comopnents u, v, w, solution (2.1) correspodns to a flow

$$u = u_0(r), \ v = 0, \ w = w_0(r); \ au_0 = brw_0, \tag{2.2}$$

specified by one arbitrary function $u_0(r)$ or $w_0(r)$. For flows with rotational symmetry ($a = 0$) the equivalents of the hydrostatic states will be flows with circular flow lines

$$u = u_0(r),\ v = w = 0, \tag{2.3}$$

where the function $u_0(r)$ on the interval of Eq. (1.8) is specified arbitrarily.

Now let

$$\alpha = \alpha(r, \mu, t),\ v = v(r, \mu, t),\ \beta = \beta(r, \mu, t) \tag{2.4}$$

be an exact nonsteady state solution of Eqs. (1.4)-(1.7), considered as a perturbation of a "rest state", Eq. (2.1). Then

Statement 1. Over the entire flow region τ of Eq. (2.1) let the inequality

$$0 \leqslant c^- \leqslant g/(\beta_0^2)_r \leqslant c^+ < \infty \tag{2.5}$$

with constants c^- and c^+ be satisfied. Then perturbations α, v, $\sigma \equiv \beta^2 - \beta_0^2$ of the flow of Eq. (2.1) can be evaluated in terms of their initial values α_*, v_*, σ_* in the following manner:

$$\int_\tau \left(\frac{\alpha^2}{R} + v^2 + c^-\sigma^2\right) d\tau \leqslant \int_\tau \left(\frac{\alpha_*^2}{R} + v_*^2 + c^+\sigma_*^2\right) d\tau. \tag{2.6}$$

Proof. We will use the notation $\rho \equiv \beta^2$, $\rho_0 \equiv \beta_0^2$. From Eqs. (1.9), (1.10) we compose a conservative functional

$$F(\alpha, v, \rho) = \int_\tau \left[\frac{\alpha^2}{R} + v^2 + \rho U + \Phi(\rho)\right] d\tau,$$

which can be represented in the form of three terms:

$$F(\alpha, v, \rho) \equiv F(0, 0, \rho_0) + F_1 + F_2,$$
$$F_1 \equiv \int_\tau \sigma[\varphi(\rho_0) + \Phi'(\rho_0)]\, d\tau,$$
$$F_2 \equiv \int_\tau \left[\frac{1}{2}\left(\frac{\alpha^2}{R} + v^2\right) + \Phi(\rho_0 + \sigma) - \Phi(\rho_0) - \Phi'(\rho_0)\sigma\right] d\tau.$$

where a prime indicates differentiation with respect to the argument. In F_1 the function $U(r)$ is replaced by $U = \phi(\rho_0)$, obtained by eliminating r from $U = U(r)$, Eq. (1.9) and $\rho = \rho_0(r)$, Eq. (2.1). In light of Eq. (2.5) the function $\phi(\rho_0)$ is monotonic. Making use of the arbitrariness of $\Phi(\rho)$, we take $\Phi'(\rho_0) \equiv -\phi(\rho_0)$, after which it develops that $F_1 \equiv 0$, and the function F_2 is time-independent.

Further, since

$$\varphi'(\rho_0) = \frac{dU}{dr}\Big/\frac{d\rho_0}{dr} = -g/(\beta_0^2)_r,$$

Eq. (2.5) is equivalent to the inequality

$$c^- \leqslant \Phi'' \leqslant c^+, \tag{2.7}$$

which is satisfied over the range of ρ_0 in the region τ. Let the function $\Phi(\rho)$ be defined for all other ρ values with conservation of the property of Eq. (2.7). Then for any numbers h and ℓ, by integrating (2.7) one can obtain

$$(1/2)c^- l^2 \leqslant \Phi(h + l) - \Phi(h) - \Phi'(h)l \leqslant (1/2)c^+ l^2. \tag{2.8}$$

Now with conservation of F_2, Eq. (2.6) follows from Eq. (2.8).

In the presence of rotational asymmetry ($a = 0$) Eq. (2.6) for perturbations of the flow, Eqs. (2.3), (1.8), reduces to the form

$$\int_{R_1}^{R_2} (v^2 + w^2 + c^-\sigma^2)\, r dr \leqslant \int_{R_1}^{R_2} (v_*^2 + w_*^2 + c^+\sigma_*^2)\, r dr, \tag{2.9}$$

where $\sigma \equiv r^2(u^2 - u_0^2)$; condition (2.5) gives

$$c^- \leqslant [r^3 (r^2 u_0^2)_r]^{-1} \leqslant c^+. \quad (2.10)$$

The final constant c^+ exists only for $R_1 > 0$. At $R_1 = 0$ the formulation of the estimate must be changed for $u_0(r)$ values of practical interest.

Estimates of the perturbations somewhat more accurate than Eq. (2.6) can be obtained in the two following cases.

Statement 2. If over the entire flow region τ of Eq. (2.1) we have

$$0 \leqslant g/(\beta_0^2)_r = \text{const} < \infty, \quad (2.11)$$

then for any perturbations of Eq. (2.4) the integral

$$\int_\tau \left\{\frac{\alpha^2}{R} + v^2 + [g/(\beta_0^2)_r]\,\sigma^2\right\} d\tau = \text{const} \quad (2.12)$$

is time-independent. This can be proved from the coincidence of functional (2.12) with F_2 given condition (2.11).

For perturbations with infinitely small amplitude direct calculations will verify

Statement 3. If over the entire flow region τ of Eq. (2.1) we have

$$0 \leqslant g/(\beta_0^2)_r < \infty, \quad (2.13)$$

then the integral of Eq. (2.12) is time-independent, with α, v and σ corresponding to the solution of the problem of Eqs. (1.4)-(1.7) linearized to Eq. (2.1).

The presence of upper limits for an arbitrary perturbation in terms of its initial value, Eqs. (2.6), (2.9), (2.12) implies stability in the mean square solution Eqs. (2.1)-(2.3) in the sense of Lyapunov's definition [5, 6].

The estimate of Eqs. (2.9), (2.10) is a nonlinear variant of the Rayleigh stability criterion [1, 7], widely known in linear stability theory, which guarantees stability of a flow with rotational symmetry, Eq. (2.3), relative to perturbations with rotational symmetry given the condition of increase in the square of the circulation $r^2u_0^2$ with increase in radius r. Statements 1 and 2 provide nonlinear analogs to the Rayleigh criterion for flows of Eq. (2.2) with more complex (helical) geometry.

3. Analogs of Planar Motion of a Homogeneous Liquid. In light of the equation $D\beta = 0$ solutions of system (1.4) with β = const are an independent class. Specification of initial data in this class guarantees the applicability of the solutions thereto. The order of system (1.4) with respect to time then decreases by one, and it becomes a first order system. After replacement of the first two equations of Eq. (1.4) with the consequences of Eq. (1.5) we obtain a system

$$D\lambda = 0,\ \lambda \equiv \omega + 2ab\beta/R,\ v_r + v/r + \alpha_\mu/r = 0, \quad (3.1)$$

which upon introduction of the flow function $\psi(rv = -\psi_\mu,\ \alpha = \psi_r)$ reduces to a single equation in ψ. The boundary condition Eq. (1.7) reduces to ψ = const on Eq. (1.6), while the value of the constant may differ on each component of the boundary.

The integral I, Eq. (1.10), is meaningless for this class of motions. At the same time there exists another integral valid only for motions with β = const and definable in terms of an arbitrary function $\Phi(\lambda)$:

$$J = \int \Phi(\lambda)\, d\tau = \text{const}. \quad (3.2)$$

In the special case $\Phi(\lambda) \equiv \lambda$ conversion of the integral J to a surface integral over $\partial\tau$ gives an analog to conservation of velocity circulation along the flow boundary:

$$\Gamma[\psi] \equiv \oint_{\partial\tau} \left(\frac{\psi_r}{R}\, r d\mu - \frac{\psi_\mu}{r}\, dr\right) = \text{const}. \quad (3.3)$$

It follows directly from the equations of motion Eq. (1.4) that if the boundary $\partial\tau$ consists of two closed components (the region τ is double-bounded), then the value of the integral (3.3) is conserved separately on each component.

Let some steady state solution of Eqs. (3.1), (1.7) be specified

$$\psi = \psi_0(r, \mu),\ \beta = \beta_0 = \text{const},\ \lambda = \lambda_0(r, \mu, \beta_0). \tag{3.4}$$

From the equation $D\lambda = 0$ it follows that there exists a functional dependence $\psi_0 = \Psi(\lambda_0)$. Further, let $\psi(r, \mu, t) = \psi_0 + \varphi(r, \mu, t)$, $\lambda(r, \mu, t) = \lambda_0 + \varkappa(r, \mu, t)$ be some nonsteady state solution of Eqs. (3.1), (1.7) which can be considered a perturbation of the flow of Eq. (3.4). At $t = 0$ on each component of the boundary $\partial\tau$ we assume that $\Gamma[\phi] = 0$, then in view of Eq. (3.3) on the boundaries $\Gamma[\phi] = 0$ at any moment in time. Then

Statement 4. Over the entire region τ of the flow of Eq. (3.4) let the inequality

$$0 \leqslant c^- \leqslant d\Psi/d\lambda_0 \leqslant c^+ < \infty \tag{3.5}$$

with constants c^- and c^+ be satisfied. Then perturbations ϕ, κ can be evaluated in terms of their initial values ϕ_*, κ_* in the following manner:

$$\int_\tau \left(\frac{\varphi_r^2}{R} + \frac{\varphi_\mu^2}{r^2} + c^-\varkappa^2\right) d\tau \leqslant \int_\tau \left(\frac{\varphi_{*r}^2}{R} + \frac{\varphi_{*\mu}^2}{r^2} + c^+\varkappa_*^2\right) d\tau. \tag{3.6}$$

The proof of this statement repeats that of statement 1 with slight changes and will not be presented here. It will be sufficient to note that the integrals E, Eq. (1.9), and J, Eq. (3.2), are used.

The imposition of the condition $\Gamma[\phi] = 0$ on the perturbation field is not physically limiting, since change in Γ is equivalent to transition from Eq. (3.4) to some other nearby steady state solution, the stability of which will then be investigated.

The result obtained can be significantly expanded in the important cases of solutions of Eq. (3.4) with rotational symmetry and circular boundary geometry, Eq. (1.8). In terms of the original velocity components u, v, w such flows are specified by two related functions of r:

$$u = u_0(r),\ v = 0,\ w = w_0(r);\ bru_0 + aw_0 = \beta_0. \tag{3.7}$$

Such flows and nonflow conditions on the boudnary, Eq. (1.8), are invariant relative to shifts along the axis of symmetry (z axis). This permits consideration of the stability problem in any coordinate system moving relative to the original system along the z axis with constant velocity M. As a result, we obtain the same statement 4, with inequality (3.5) taking on the more concrete form

$$c^- \leqslant \frac{d\Psi}{d\lambda_0} = \frac{d\psi_0}{dr} \Big/ \frac{d\lambda_0}{dr} = \frac{\alpha_0 + brM}{A} \leqslant c^+, \tag{3.8}$$

where $\alpha_0 \equiv au_0 - brw_0$. The quantity $A \equiv d\lambda_0/dr$ is independent of and can be specified by any of three expressions

$$A \equiv \left[\frac{1}{r}\left(\frac{r\alpha_0}{R}\right)_r + \frac{2ab\beta_0}{R^2}\right]_r = \left[\frac{(ru_0)_r}{ar}\right]_r = -\left(\frac{w_{0r}}{br}\right)_r. \tag{3.9}$$

We may also add here the same inequality (3.8) may be obtained by transition into a coordinate system uniformly rotating about the z axis rather than one in translation. Thus we have

Statement 5. If there exists a constant M such that over the entire interval of Eq. (1.8) Eq. (3.8) is satisfied, then the flow of Eq. (3.7) is stable in the sense of the presence of Eq. (3.6).

In particular, this stability condition contains the narrower

Statement 6. If the continuous function A(r), Eq. (3.9), has no zeroes within the interval of Eq. (1.8), then the folow of Eq. (3.7) is stable.

The possibility of choosing a constant M at $A \neq 0$ for a flow between two cylinders is obvious. For a flow within a tube ($R_1 = 0$ in Eq. (1.8)) it is sufficient to note that in the absence of singularities in the vorticity distribution for $r \to 0$ it is always true that $\alpha_0(r) \sim r^n$ with the constant $n \geq 1$.

Statements 3-6 are a generalization to a new class of motions, Eq. (1.1), and to finite amplitude perturbations of Rayleigh's result widely known in linear stability theory [1] concerning stability of planeparallel flow in the absence of an inflection point in the velocity profile. In the special case (b = 0) motions of Eq. (1.1) are planar and statements 3-6 give the previously known results of Rayleigh [1], Fjortoft [7], and Arnold [2, 8]. At a = 0 (stability of an axisymmetric flow in a circular tube) the linear variant of statement 6 was also obtained by Rayleigh [9].

4. Rotating Flows with Translational Symmetry. We will consider motion of a liquid with homogeneous density in a coordinate system rotating with a constant velocity $\Omega/2$. The equations of motion can be written in the form [6]

$$D\mathbf{u} + \mathbf{\Omega}\times\mathbf{u} = -\nabla p^*,\ \operatorname{div}\mathbf{u} = 0,\ D \equiv \partial/\partial t + \mathbf{u}\cdot\nabla. \tag{4.1}$$

Here u is the velocity vector; p* is the modified pressure, including a "centrifugal" addition.

Let k be a unit vector specifying a fixed (in the rotating system) direction and forming with the vector Ω an angle $\theta (0 \leq \theta \leq \pi)$. We will study the class of solutions of Eq. (4.1) in which u and p* do not change along the direction of the vector k. We introduce a Cartesian coordinate system x, y, z, such that the z axis is parallel to the vector k, and the vector Ω lies in the plane xz. For the motions considered the velocity field u = (u, v, w) and the pressure p* are independent of the coordinate z:

$$\mathbf{u} = \mathbf{u}(x, y, t),\ p^* = p^*(x, y, t). \tag{4.2}$$

After introducing the notation of [3]

$$\mathbf{\Omega} = (\Omega_1, 0, \Omega_3),\ \rho \equiv w - \Omega_1 y,\ \mathbf{g} = \mathbf{k}\times\mathbf{\Omega} = (0, g, 0),\ g \equiv \Omega_1 \tag{4.3}$$

system (4.1) for motions of Eq. (4.2) can be transformed to

$$\begin{gathered} Du = -p_x,\ Dv = -p_y + \rho g, \\ D\rho = 0,\quad u_x + v_y = 0,\quad D \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \end{gathered} \tag{4.4}$$

where $p \equiv p^* - \Omega_3\psi + \frac{1}{2}\Omega_1^2 y^2$; ψ is the flow function for which $u = -\psi_y$, $v = \psi_x$.

If the motion of Eq. (4.2) takes place in a fixed region, then its boundary must have the form of a cylindrical surface with directrix parallel to the z axis, i.e., the expression

$$f(x, y) = 0 \tag{4.5}$$

is specified. In the plane xy curve (4.5) limits the flow region τ, which may also not be single-bound. Its boundary $\partial\tau$, Eq. (4.5) may either by closed or extend to infinity. The nonflow condition on the boudnary of Eq. (4.5) gives

$$uf_x + vf_y = 0. \tag{4.6}$$

The coincidence of Eqs. (4.4)-(4.6) with the equations and corresponding boundary conditions for planar motions of inhomogeneous density (stratified) liquids in the Boussinesq approximation [10] is remarkable. Because of this, all results valid for planar flows of stratified liquids are valid for rotating flows with translational symmetry. In particular, the kinetic energy integral for Eqs. (4.1), (4.6) in terms of Eqs. (4.3), (4.4) can be written in the form of a sum of fictitious "kinetic" T and "potential" P energies:

$$\begin{gathered} E = T + \Pi = \text{const}, \\ T \equiv \frac{1}{2}\int_\tau (u^2 + v^2)\,d\tau,\quad \Pi \equiv \int_\tau \rho U\,d\tau \quad (d\tau \equiv dx\,dy), \end{gathered} \tag{4.7}$$

where U is the potential, introduced in accordance with $g = -\nabla U$. The other integral (4.4), (4.6) is

$$I = \int_\tau \Phi(\rho)\, d\tau \qquad (4.8)$$

with arbitrary function $\Phi(\rho)$ (compare Eqs. (1.9), (1.10)).

The analogs to hydrostatic equilibrium states of the class of Eq. (4.2) are exact solutions of Eq. (4.4) of form

$$u = v = 0, \ \rho = \rho_0(y). \qquad (4.9)$$

In the original terms of Eq. (4.1) Eq. (4.9) describes a shear flow in one direction (plane-parallel flow):

$$u = v = 0, \ w = w_0(y). \qquad (4.10)$$

The functions $\rho_0(y)$ and $w_0(y)$ in Eqs. (4.9), (4.10) are arbitrary.

Now let

$$u = u(x, y, t), \ v = v(x, y, t), \ \rho = \rho_0(y) + \sigma(x, y, t)$$

be some exact nonsteady state solution of Eqs. (4.4)-(4.6), considered as a perturbation of "state of rest", Eq. (4.9). Then

Statement 7. Over the entire flow region τ let the inequality

$$0 \leqslant c^- \leqslant g/\rho_{0y} \leqslant c^+ < \infty$$

with constants c^- and c^+ be satisfied. Then perturbations u, v, σ of the flow of Eqs. (4.9), (4.10) can be evaluated in terms of their initial values in the following manner:

$$\int_\tau (u^2 + v^2 + c^-\sigma^2)\, d\tau \leqslant \int_\tau (u_*^2 + v_*^2 + c^+\sigma_*^2)\, d\tau.$$

The proof of this statement is based on the presence of integrals E, Eq. (4.7), and I, Eq. (4.8), and almost literally repeats the proof of statement 1, so will not be presented.

For the interest case of linear functions $w_0(y)$ or $\rho_0(y)$ (Couette profile in the gap between planes) there is an analog statement 2.

Statement 8. Over the entire region τ let $0 \le g/\rho_{0y} = \text{const} < \infty$, then for the problem of Eqs. (4.4)-(4.6) the solution of Eqs. (4.9), (4.10) is stable in the sense of time independence of the integral

$$\int_\tau [u^2 + v^2 + (g/\rho_{0y})\sigma^2]\, d\tau = \text{const}.$$

For the problem of Eqs. (4.4)-(4.6) there also exists a class of motions with ρ = const (see Section 3), which reduces to planar motions of a homogeneous liquid with $\Omega = 0$. Such motions are a special case of those studied in Section 3 (with b = 0), and were considered previously in [2, 8]. Thus their special study would be superfluous here.

5. General Remarks. In conclusion it is fitting to consider several points important for an overall understanding of the questions touched.

1. From the mathematical viewpoint all the statements derived have the character of a priori estimates, since corresponding theorems for the existence of solutions were not proved.

2. Statements 1 and 4 were proved by the method of "coupling" of integrals [5, 6] in the form of [2, 7, 8].

3. To clarify the relationship between statements 1 and 4 by variation principles, it is sufficient to note that vanishing of the functional F_1 given proof of statement 1 is equivalent to the presence of an absolute extremum in F in the flows of Eqs. (2.1), (2.2). Similarly the functional E + J, Eqs. (1.9), (3.2) has an absolute extremum for any flow Eq. (3.4). Imposition of the conditions (2.3), (3.5) guarantees that both of these extrema are minima. The term "absolute extremum" is used here in the sense that in the class of motions

with a given symmetry no other additional limitations on variations of the hydrodynamic fields are imposed.

4. Considering the similarity of Eqs. (1.1), (4.2) to motions of a stratified liquid discussed in Sections 1, 4, statements 1-3 and 7 are analogs of the Lagrange theorem of analytical mechanics [5, 6] regarding stability of the rest state of a mechanical system having an isolated minimum in potential energy. It can be proved, for example, that given conditions (2.5), (2.10), (2.11), (2.13) the fictitious potential energy P, Eq. (1.9), has an isolated minimum for variations of the "density" β which satisfy the condition I = const, Eq. (1.10).

5. The evaluations obtained, which indicate mean square stability, may be unsatisfactory for some purposes. In fact, if we measure deviations of the solutions not with mean, but rather maximum perturbation values, then as Lyapunov noted, conservation of energy proves insufficient for proof of the statements regarding stability. To obtain corresponding evaluations it is necessary to introduce additional limitations on the solution, with the problem of justifying these limitations remaining open [6].

6. All statements on stability presented herein are arbitrary in the sense that stability is guaranteed only for special classes of perturbations having the same symmetry as the main flow. Naturally proof of stability in such classes is of limited physical significance. However the difficulties in studying exact nonlinear hydrodynamics problems are so great that even information on the properties of special classes of motion are of interest.

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